International Journal of Theoretical Physics, Vol. 30, No. 10, 1991

Normal Functionals and Spaces of Weights

J. Hagler[1] and C. Schindler[1]

Received June 6, 1990

It is shown that the cone of positive normal functionals on a hypergraph $\mathscr{A}$ may be substantially larger than the cone of positive weights on $\mathscr{A}$. For a semiclassical hypergraph $\mathscr{A}$ the two cones coincide if and only if the number of edges of $\mathscr{A}$ of cardinality ≥ 2 is finite. This disproves an earlier statement of T. A. Cook.

1. INTRODUCTION

Cook (1985) raised the question of whether the positive weights on a hypergraph $\mathscr{A}$ coincide with the positive normal functionals in the bidual of $\mathscr{J}(\mathscr{A})$, the space of Jordan weights on $\mathscr{A}$. (Unexplained notions are discussed below.) He suggested [Proposition 9 of Cook (1985)] that the answer is positive. In this note we reconsider this problem and show (see Example 4 and Theorem 2) that this proposition is false.

This note is organized as follows: We briefly describe cases (Examples 1–3) where the answer to Cook's question is affirmative. These examples suggest a place to look for a counterexample (Example 4). In this example we construct a nonreflexive ordered Banach space for which all functionals in the bidual are normal.

From the existence of this example it follows (Theorems 1 and 2) that there is a large class of ordered Banach spaces V for which the subspace of normal functionals in the bidual of V does not coincide with V. In particular, Theorem 2 states that the space of Jordan weights on a semiclassical hypergraph (a hypergraph whose edges are pairwise disjoint) coincides with the space of normal functionals if and only if the number of edges of cardinality ≥ 2 is finite.

[1]Department of Mathematics and Computer Science, University of Denver, Denver, Colorado 80208.

0020-7748/91/1000-1337$06.50/0

2. NORMAL FUNCTIONALS

Let V be a Banach space with a partial order $\leq$. The Banach space dual and bidual of V will be denoted by V^* and V^{**}, respectively. The cone of positive elements in V will be denoted by V_+; similarly, $V_+^* = \{f \in V^* : f(V_+) \subset \mathbf{R}_+\}$, and $V_+^{**} = \{\phi \in V^{**} : \phi(V_+^*) \subset \mathbf{R}_+\}$. Elements of V_+, V_+^*, or V_+^{**} will be called *positive.*

A net $(f_\delta)_{\delta \in D}$ in V^* is said to be increasing if whenever $\delta_1 \leq \delta_2$ $(\delta_1, \delta_2 \in D)$, then $f_{\delta_2} - f_{\delta_1} \in V_+^*$. We say that a linear functional $\phi \in V_+^{**}$ is *normal* if for every increasing net $(f_\delta)_{\delta \in D}$ in V^*, converging weak* to f, $\phi(f) = \lim_{\delta \in D} \phi(f_\delta)$.

By V_N^{**} (respectively $V_{+,N}^{**}$) we denote the subset of V^{**} of normal (respectively positive normal) functionals. It is easy to see that V_N^{**} is a linear subspace of V^{**} and that V_+^{**} is a positive subset of V^{**}.

Let $\chi: V \to V^{**}$ denote the canonical embedding of V into V^{**}. We will identify V and V_+ with their images $\chi(V)$ and $\chi(V_+)$ in V^{**}. Since χ preserves positivity and since every $x \in V$ is weak* continuous on V^*, we have that $V \subset V_N^{**}$ and $V_+ \subset V_{+,N}^{**}$.

Each ordered Banach space V occurring in our examples will be a base norm space (Alfsen, 1971; Asimov and Ellis, 1980), i.e., a space whose unit ball coincides with the absolute convex hull of the set $\{x \in V_+ : \|x\| = 1\}$.

Example 1. If $V = l_1(X)$ for some nonempty set X, then $V = V_N^{**}$.

Recall that $l_1(X) = \{f: X \to \mathbf{R} : \|f\| = \sum_{x \in X} |f(x)| < \infty\}$. The canonical linear order $\leq$ on $l_1(X)$ is the pointwise order, i.e., $f \leq g \Leftrightarrow f(x) \leq g(x)$ for all $x \in X$. It is well known that V^* is isometrically isomorphic to $l_\infty(X)$ and that V^{**} is isometrically isomorphic to the space of bounded, finitely additive measures on 2^X.

To see that $V = V_N^{**}$, let $\phi \in V_N^{**}$. Since $\phi = \phi_1 + \phi_2$, where $\phi_1 \in V$ and ϕ_2 is purely finitely additive [i.e., $\phi_2(F) = 0$ if F is a finite subset of X], it suffices to show that every purely finitely additive normal measure ϕ is the 0-measure.

Let $B \subset X$ be infinite, and $\mathscr{D}$ the directed set of finite subsets of B, ordered by set inclusion. Then $(1_F)_{F \in \mathscr{D}}$ is an increasing net in $l_\infty(X)$ that converges weak* to 1_B. Since ϕ is normal, $\phi(B) = \lim_{F \in \mathscr{D}} \phi(F) = 0$. Therefore, $\phi = 0$, and the assertion is proved. ■

Example 2. Let A be a von Neumann algebra and A_{sa} its self-adjoint part. The positive elements in A_{sa} form a generating cone in A_{sa}. Let V denote the ordered Banach space predual of A_{sa}, so that $V^* = A_{sa}$. It is a standard result in the theory of von Neumann algebras (Pedersen, 1979; Sakai, 1971) that $V_N^{**} = V$. In particular, these results apply to the space $\mathscr{B}(H)$ of bounded operators on a complex Hilbert space H, which is a von Neumann algebra whose predual is the space of trace-class operators on H.

3. SPACES OF WEIGHTS

A *hypergraph* (or *quasimanual*) is a pair $\mathscr{A}=(X, \mathcal{O})$, where X is a non-empty set and $\mathcal{O}$ is a covering of X by nonempty subsets of X. The elements of X and $\mathcal{O}$ are called *vertices* and *edges*, respectively, of $\mathscr{A}$. A *positive weight* on the hypergraph $\mathscr{A}$ is a map $\mu: X \to \mathbf{R}_+$ satisfying $\sum_{x\in E} \mu(x) < \infty$ for all $E \in \mathcal{O}$ and

$$\sum_{x\in E} \mu(x) = \sum_{x\in F} \mu(x) \qquad \text{for all} \quad E, F \in \mathcal{O}$$

Let $\|\mu\|$ denote the common value of $\sum_{x\in E} \mu(x)$, $E\in\mathcal{O}$.

Let $K(\mathscr{A})$ denote the set of all positive weights on $\mathscr{A}$, and $\mathscr{J}(\mathscr{A}) = \{\nu-\mu: \mu, \nu\in K(\mathscr{A})\}$. Elements of $\mathscr{J}(\mathscr{A})$ are called *Jordan weights* on $\mathscr{A}$. Let $\Omega(\mathscr{A}) = \{\mu\in K(\mathscr{A}): \|\mu\|=1\}$ be the *probability weights* on $\mathscr{A}$. A natural partial order on $\Omega(\mathscr{A})$ is given by $\mu \le \nu \Leftrightarrow \mu(x) \le \nu(x)$ for all $x\in X$. For every $\lambda \in \mathscr{J}(\mathscr{A})$, define

$$\|\lambda\|_B = \inf\{\|\rho\| + \|\sigma\| : \lambda = \sigma - \rho,\ \rho,\ \sigma\in K(\mathscr{A})\}$$

Clearly, $\|\mu\|_B = \|\mu\|$ for all $\mu \in K(\mathscr{A})$. It is easy to verify that $\|\cdot\|_B$ is a norm on $\mathscr{J}(\mathscr{A})$ (*base norm*) and that the $\|\cdot\|_B$-unit ball in $\mathscr{J}(\mathscr{A})$ is equal to the set $conv(\Omega(\mathscr{A}) \cup -\Omega(\mathscr{A}))$.

Cook (1985) showed that $(\mathscr{J}(\mathscr{A}), \|\cdot\|_B)$ is a Banach space and that every bounded increasing net $(f_\delta)_{\delta\in D}$ in $\mathscr{J}(\mathscr{A})^*$ converges weak* to its least upper bound in $(\mathscr{J}(\mathscr{A})^*, \le)$, the ordered dual of $\mathscr{J}(\mathscr{A})$.

A hypergraph $\mathscr{A}=(X, \mathcal{O})$ is said to be *classical* if $\mathcal{O}=\{X\}$ and *semiclassical* if $\mathcal{O}$ is a partition of X. Note that for every classical hypergraph $\mathscr{A}=(X, \{X\})$, the ordered Banach space $\mathscr{J}(\mathscr{A})$ is identical to the space $(l_1(X), \|\cdot\|, \le)$ discussed in Example 1. For a classical hypergraph $\mathscr{A}$ we thus have $\mathscr{J}(\mathscr{A}) = \mathscr{J}(\mathscr{A})_N^{**}$.

Another norm on $\mathscr{J}(\mathscr{A})$ is the *variation norm*, defined by

$$\|\lambda\|_v = \sup\left\{\sum_{x\in E} |\lambda(x)| : E\in\mathcal{O}\right\}$$

For semiclassical hypergraphs, the norms $\|\cdot\|_B$ and $\|\cdot\|_v$ coincide, so we omit the subscript and refer to $\|\cdot\|$. [Note, however, that there exist hypergraphs $\mathscr{A}$ for which $(\mathscr{J}(\mathscr{A}), \|\cdot\|_v)$ is not complete (Schindler, 1986).]

Example 3. Every Hilbert space H gives rise to a hypergraph as follows: X is the collection of all one-dimensional subspaces of H and $\mathcal{O}$ is the collection of all maximal orthogonal subsets of X. [The pair $\mathscr{A}_H=(X, \mathcal{O})$ is called the *frame manual* of H.] For Hilbert spaces of dimension $\neq 2$, Gleason's theorem (Gleason, 1957) provides an isometric order isomorphism from

$\mathscr{J}(\mathscr{A}_H)$ onto the ordered Banach space V of self-adjoint trace class operators on H.

Thus, according to Example 2, $\mathscr{J}(\mathscr{A}_H)_N^{**} = \mathscr{J}(\mathscr{A}_H)$. If $\dim(H) = 2$, then Gleason's theorem does not apply, and, as the following example shows, $\mathscr{J}(\mathscr{A}_H)_N^{**} = \mathscr{J}(\mathscr{A}_H)^{**}$.

Example 4. Let A be an arbitrary nonempty set,

$$X = \{(x, i) : x \in A, i = 1, 2\}$$

Letting $E_x = \{(x, 1), (x, 2)\}$ for every $x \in A$, and $\mathscr{O} = \{E_x : x \in A\}$, we obtain a semiclassical hypergraph $\mathscr{A} = (X, \mathscr{O})$.

We claim that every weak* convergent increasing net $(f_\delta)_{\delta \in D}$ in $\mathscr{J}(\mathscr{A})^*$ is norm convergent. From this, it is immediate that $\mathscr{J}(\mathscr{A})_N^{**} = \mathscr{J}(\mathscr{A})^{**}$.

Proof. Observe that the net $(f_\delta)_{\delta \in D}$ increases and converges to f weak* (in norm) $\Leftrightarrow$ the net $(f - f_\delta)_{\delta \in D}$ decreases and converges to 0 weak* (in norm). Thus, it suffices to show that if $(f_\delta)_{\delta \in D}$ is a decreasing net, converging weak* to 0, then $(f_\delta)_{\delta \in D}$ converges in norm to 0.

Let $\rho \in K(\mathscr{A})$ be defined by $\rho(x, i) = 1$ for $i = 1, 2$. Then, if $\lambda \in \mathscr{J}(\mathscr{A})$, $\|\lambda\| \leq 1$, we have $-\rho \leq \lambda \leq \rho$. Let $\varepsilon > 0$. There exists $\delta_0 \in D$ such that $|f_\delta(\rho)| < \varepsilon$ for all $\delta \geq \delta_0$. Thus, if $\lambda \in \mathscr{J}(\mathscr{A})$, $\|\lambda\| \leq 1$, we have

$$-\varepsilon < f_\delta(-\rho) \leq f_\delta(\lambda) \leq f_\delta(\rho) < \varepsilon$$

for $\delta \geq \delta_0$. So, $\|f_\delta\| < \varepsilon$ if $\delta \geq \delta_0$, and we conclude that $(f_\delta)_{\delta \in D}$ converges to 0 in norm. This proves the claim.

To show that this example disproves Cook's statement, we must show that $\mathscr{J}(\mathscr{A})$ is nonreflexive whenever A is infinite. Let

$$\mathscr{U} = \{\mu \in \mathscr{J}(\mathscr{A}) : \mu(x, 1) + \mu(x, 2) = 0 \ \forall x \in A\}$$

For each $\phi \in l_\infty(A)$ define $T(\phi)$ to be the unique element $\mu \in \mathscr{U}$ satisfying $\mu(x, 1) = \phi(x)$ for all $x \in X$. The map T establishes a linear homeomorphism between $l_\infty(A)$ and the closed subspace $\mathscr{U}$ of $\mathscr{J}(\mathscr{A})$. Thus, $\mathscr{J}(\mathscr{A})$ is nonreflexive since it contains a nonreflexive subspace, $\mathscr{U}$. Since $\mathscr{J}(\mathscr{A})_+^{**}$ is a generating cone in $\mathscr{J}(\mathscr{A})^{**}$, we have that $\mathscr{J}(\mathscr{A})_{+,N}^{**} - \mathscr{J}(\mathscr{A})_+ \neq \varnothing$ whenever A is infinite. ■

We are now in a position to state positive results which follow from our examples.

Theorem 1. Let U and V be ordered Banach spaces and suppose that there exists an order-preserving linear operator $T: U \to V$ mapping U homeomorphically into V. If $U_N^{**} - U \neq \varnothing$ (resp. $U_{+,N}^{**} - U_+ \neq \varnothing$), then $V_N^{**} - V \neq \varnothing$ (resp. $V_{+,N}^{**} - V_+ \neq \varnothing$).

The proof follows easily since the adjoint operators T^* and T^{**} are weak* continuous and positive. The key step in the proof is this observation, which is a straightforward consequence of the Hahn–Banach theorem: Let U and V be Banach spaces. If $T: U \to V$ is a linear homeomorphism between U and a subspace of V, and $T^{**}: U^{**} \to V^{**}$ is its second adjoint, then $T^{**}(\phi) \in V \Leftrightarrow \phi \in U$.

We can now prove our main result.

Theorem 2. Let $\mathscr{A} = (X, \mathscr{O})$ be a semiclassical hypergraph with $|E| \geq 2$ for all $E \in \mathscr{O}$. Then the following statements are equivalent:

(i) $\mathscr{J}(\mathscr{A})^{**}_{+,N} = \mathscr{J}(\mathscr{A})_+$.

(ii) $\mathscr{O}$ is finite.

Proof. Assume that $\mathscr{O} = \{E_i : i \in I\}$ is infinite. For each $i \in I$, let $x_{i,1}$ and $x_{i,2}$ be distinct elements of E_i. Following the procedure of Example 4, construct a semiclassical hypergraph $\mathscr{B} = (Y, \mathscr{P})$ from the set I (i.e., $Y = I \times \{1, 2\}$, $E_i = \{(i, 1), (i, 2)\}$ for $i \in I$, and $\mathscr{P} = \{E_i : i \in I\}$). For every $\mu \in \mathscr{J}(\mathscr{B})$ define $T(\mu)$ to be the unique element of $\mathscr{J}(\mathscr{A})$ satisfying $T(\mu)(x_{i,1}) = \mu(i, 1)$ and $T(\mu)(x_{i,2}) = \mu(i, 2)$ for all $i \in I$, and $T(\mu)(y) = 0$ for all $y \in \bigcup_{i \in I} (E_i - \{x_{i,1}, x_{i,2}\})$. It is easy to see that the map $T: \mathscr{J}(\mathscr{B}) \to \mathscr{J}(\mathscr{A})$ is order preserving and embeds $\mathscr{J}(\mathscr{B})$ isometrically into $\mathscr{J}(\mathscr{A})$. From Theorem 1 and the fact that $\mathscr{J}(\mathscr{B})^{**}_{+,N} - \mathscr{J}(\mathscr{B})_+ \neq \varnothing$ it follows that $\mathscr{J}(\mathscr{A})^{**}_{+,N} - \mathscr{J}(\mathscr{A})_+ \neq \varnothing$. This proves that (i) $\Rightarrow$ (ii).

Assume now that $\mathscr{O}$ is finite. Then every $\mu \in \mathscr{J}(\mathscr{A})$ is also a Jordan weight on the classical hypergraph $\mathscr{B} = (X, \{X\})$. Let $\|\cdot\|_{\mathscr{A}}$ and $\|\cdot\|_{\mathscr{B}}$ denote the norms in $\mathscr{J}(\mathscr{A})$ and $\mathscr{J}(\mathscr{B})$, respectively, and M the cardinality of $\mathscr{O}$. Then $\|\mu\|_{\mathscr{A}} \leq \|\mu\|_{\mathscr{B}} \leq M\|\mu\|_{\mathscr{A}}$ holds for all $\mu \in \mathscr{J}(\mathscr{A})$. Hence, $\mathscr{J}(\mathscr{A})$ is homeomorphically and order isomorphically embedded in $\mathscr{J}(\mathscr{B})$. If we had

$$\mathscr{J}(\mathscr{A})_+ \neq \mathscr{J}(\mathscr{A})^{**}_{+,N}$$

then Theorem 1 would imply that $\mathscr{J}(\mathscr{B})_+ \neq \mathscr{J}(\mathscr{B})^{**}_{+,N}$. But this contradicts the fact that for classical hypergraphs, the space of Jordan weights coincides with the space of normal functionals. Hence, $\mathscr{J}(\mathscr{A})_+ = \mathscr{J}(\mathscr{A})^{**}_{+,N}$. ■

REFERENCES

Alfsen, E. M. (1971). *Compact Convex Sets and Boundary Integrals*, Springer, Berlin.

Asimov, L., and Ellis, A. J. (1980). *Convexity Theory and its Applications in Functional Analysis*, Academic Press, London.

Cook, T. A. (1985). *International Journal of Theoretical Physics*, **24**, 1113–1131.

Gleason, A. (1957). *Journal of Rational Mechanics and Analysis*, **6**, 885–893.

Pedersen, G. K. (1979). *C*-Algebras and Their Automorphism Groups*, Academic Press, London.

Sakai, S. (1971). *C*-Algebras and W*-Algebras*, Springer, Berlin.
Schindler, C. (1986). Decompositions of measures on orthologics, Dissertation, Universität Bern.